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THE FORM OF THE LOSS OF STABILITY OF A BOUNDARY LAYER ON A FLEXIBLE SURFACE AT HIGH REYNOLDS NUMBERS†

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The linear problem of the stability of a boundary layer on a flexible surface with respect to small two-dimensional perturbations in the form of travelling waves is considered. A boundary-layer model with self-induced pressure (a three-layer model) is used to describe the fluid flow. The flexible surface is modelled by a thin plate lying on an elastic base. It is shown that, in the system in question two forms of loss of stability are possible: Tollmien–Schlichting instability and instability caused by the action of hydrodynamic pressure on the surface around which the flow occurs. A special feature of the latter form of instability is that there is no neutral system of perturbations or a sudden onset of instability. The solutions obtained are analysed and the conditions leading to the appearance of such instability are established.

An analysis of the natural vibrations of a boundary layer and of a flexible surface over which a flow occurs is of interest in investigating the effect of the vibration of a wall on the transition from a laminar boundary layer to a turbulent boundary layer. In essence, it reduces to solving the Orr–Sommerfeld problem with modified boundary conditions reflecting the response of an elastic boundary to the action of pressure pulsations. Interest in the problem arose after the publication of experimental results [1] in which it was established that there is a reduction in frictional drag on an underwater missile which has a flexible coating of special construction.

The stability of the boundary layer on a flexible surface has been considered by many investigators. It was shown in [2–4] that, in the case of flow past a flexible boundary, three types of instability are possible: class A, class B, and Kelvin–Helmholtz instability and, moreover, the existence of the internal drag of the wall leads to destabilization of class A waves and stabilization of class B waves, while Kelvin–Helmholtz instability depends only slightly on the internal drag of the wall and arises when the effective rigidity of the elastic wall becomes too small to compensate for the action of the pressure forces. Waves of class A have been identified with Tollmien–Schlichting waves and waves of class B have been compared with the waves generated on the free surface of water by a wind. Both types of waves have also been detected in a boundary layer.

As a result of the modelling of a compliant surface in general form by specifying the ratio of the pressure on the surface to its deformation, it has been established [5] that the ability of a wall to deform itself in a direction normal to the surface affects the stability of laminar flow to a significantly greater extent than its ability to deform itself in a tangential direction. Later [6], the problem of stability was studied in a similar formulation.

It was shown in [4–6] that a flexible surface can stabilize a flow as a result both of an increase in the critical Reynolds number and a reduction in the rate of growth of perturbations, but it can also lead to the destabilization of the flow. In order to stabilize a flow, a flexible wall must have [4] a small internal drag in order to damp class A waves and, at the same time, sufficient drag to suppress class B waves, while the flexibility must not be so great as to give rise to Kelvin–Helmholtz instability. Doubt has therefore been expressed that the results of Kramer's experiments can only be explained on the basis of the linear theory of stability [4].

Subsequent papers were directed towards studying the effect of the parameters of a flexible surface on the stability of the flow in more detail (see, [7, 8]) and, also, making a thorough survey and comparison of the theoretical and experimental results [7, 9]. An attempt was made in [7, 8] to model the surface used in Kramer's experiments as closely as possible, and the effect of the parameters of the surface on the stability of the different classes of waves was investigated. The main conclusion drawn in [7, 8] is that a flexible surface can, in principle, lead to the persistence of laminar conditions but, in practice, the existence and interaction of the different types of waves in the boundary layer makes it very unlikely

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that, in particular, the unsuccessful attempts to confirm Kramer's results experimentally could be explained by this.

As a result of a study of the non-linear development of wave packets consisting of Tollmien–Schlichting waves, it has been shown [10] that the elasticity of the surface leads to the delay of the transition from laminar to turbulent conditions.

In this paper it is shown that, in addition to the known types of instability, one other type of instability is possible when a viscous fluid flows past a flexible surface: the system suddenly transfers from a stable position to an unstable position. A discontinuous form of the behaviour of the system as a consequence of the compliance of the wall has been obtained previously [10] as a consequence of the chosen model of the surface, over which the flow occurs, leading to singularities in the dispersion equation. It is shown below that discontinuous instability of the boundary layer occurs when the flexible surface itself loses stability under the action of the induced hydrodynamic pressure. Here, the instability displays the properties of both class A and class B. The conditions under which such instability occurs are determined and an explanation of the mechanism of its origin is suggested.

1. FORMULATION OF THE PROBLEM

Consider the flow of an incompressible viscous fluid at high Reynolds numbers past a semi-infinite plate with a flexible coating. The plate is planar in the unperturbed state. A Cartesian system of coordinates x^* and y^* is chosen such that the x^* axis lies in the plane of the unperturbed plate and the y^* axis is directed upwards. The leading edge of the plate corresponds to the value $x^* = -L$. We shall assume that the unperturbed fluid flow is two-dimensional and, far upstream, has a velocity U_∞ parallel to the x^* axis. We define the Reynolds number in the form $\mathrm{Re} = U_\infty L/\nu$, where ν is the kinematic viscosity of the fluid. We shall study two-dimensional perturbations in the boundary layer on the flexible surface and treat the linear stability problem in a time-dependent formulation.

We will assume that the parameters of the perturbations which arise are such that the fluid flow can be described by the three-deck asymptotic theory of a boundary layer (the theory of a boundary layer with a self-induced pressure). In this case, it is assumed that the perturbations are characterized by a wavelength $O(L\varepsilon^3)$, an amplitude $O(L\varepsilon^5)$ and a frequency $O(U_\infty/(L\varepsilon^2))$, where the small parameter $\varepsilon = \mathrm{Re}^{-1/8}$, which is characteristic of the three-deck theory, has been introduced. In order to describe transient processes, a theory has been developed in many papers of which we note [11–13] (see the review of other papers which make use of three-deck theory [14]). According to this theory, three domains with very different properties are formed when an unsteady boundary layer interacts with an outer flow [11]. The upper domain with characteristic size $y^* = O(L\varepsilon^3)$ is the domain of interaction with the outer potential flow. The middle domain ($y^* = O(L\varepsilon^4)$) corresponds to the main boundary layer in which the viscosity is also neglected but the flow is rotational. In the lower domain ($y^* = O(L\varepsilon^5)$), the viscosity plays a decisive role in forming the flow pattern. The pressure is produced in the lower domain and is transmitted across the middle domain to the outer flow. All the characteristic changes in the flow in each domain are defined in a length scale $O(L\varepsilon^3)$.

Appropriate expansions of the required values and scales of the change in the independent variables using a small parameter ε are introduced in each domain, and solutions are constructed in the upper and middle domains [11, 12]. The flow in the lower domain is described in the principal approximation by Prandtl-type equations. By obtaining a solution in the lower domain and matching the solutions in all three domains, it is possible to obtain (for example, see [15]) a description of the flow for the whole of the boundary layer.

For the purposes of the present investigation, it is sufficient to restrict ourselves solely to considering the flow in the lower domain. The independent variables are specified by the expressions

$$x^* = \varepsilon^3 Lx, \quad y^* = \varepsilon^5 Ly, \quad t^* = \varepsilon^2 Lt/U_\infty$$

where t^* is the dimensional time. The components of the velocity vector u^* and v^* and the pressure p^* are represented in the form of the expansions

$$u^* = U_\infty[\varepsilon U(x,y,t) + O(\varepsilon^2)], \quad v^* = U_\infty[\varepsilon^3 V(x,y,t) + O(\varepsilon^4)] \tag{1.1}$$

$$p^* = p^*_\infty + \rho U_\infty^2[\varepsilon^2 P(x,y,t) + O(\varepsilon^3)]$$

where p^*_∞ is the pressure in the unperturbed flow and ρ is the fluid density.

Substituting expressions (1.1) into the Navier–Stokes equations, we obtain, in the case of the principal terms in the expansions, the Prandtl-type equation

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=-\frac{\partial P}{\partial x}+\frac{\partial^2 U}{\partial y^2},\quad \frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0,\quad \frac{\partial P}{\partial y}=0 \tag{1.2}$$

The difference lies in the fact that the perturbed pressure $P(x, t)$ is not taken from the outer flow problem but is determined by the interaction of the flows in the upper and lower domains through the main domain of the boundary layer. This interaction is established by the following conditions. Matching of the solutions of the potential problem in the upper domain and the solutions in the main domain of the boundary layer leads to the condition (the condition of free interaction)

$$P(x,t)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\partial A/\partial x_1}{x-x_1}dx_1 \tag{1.3}$$

The arbitrary function $A(x, t)$ has the meaning of the magnitude of the instantaneous displacement of the stream lines in the main domain of the boundary layer.

Matching of the expansions in the lower and main domains of the boundary layer gives the limiting condition

$$U(x, y, t)\to\lambda y+\lambda A(x, t)\ \text{when}\ \ y\to\infty \tag{1.4}$$

The quantity γ is determined from the Blasius solution $U_b(y_2)$ for the unperturbed boundary layer using the formula $\lambda = dU_b(0)/dy_2$, where $y_2 = y^*/(L\varepsilon^4)$ is the vertical coordinate corresponding to the main domain of the boundary layer.

The three-deck theory actually accomplishes the link across the main domain of the boundary layer between the outer domain and the viscous domain, which is adjacent to the surface over which the flow occurs. This interaction is established by condition (1.4), which matches the lower and main domains, and relation (1.3), which matches the main domain with the outer domain of the boundary layer.

Note that the quantity γ can be eliminated from the treatment by means of a change of variables [11]; we shall therefore henceforth assume that $\gamma = 1$.

The no-slip conditions

$$U(x,\eta,t)=0,\quad V(x,\eta,t)=\frac{\partial\eta(x,t)}{\partial t} \tag{1.5}$$

where $\eta(x, t)$ is the vertical displacement of the plate surface from its unperturbed position, are satisfied on the surface over which the flow occurs and which is defined by the equation $y = \eta(x, t)$. Here, it is assumed that each point of the flexible surface executes an oscillatory motion solely in a vertical direction.

We shall model the flexible surface [7] as a thin plate lying on an elastic base, the motion of which is described by the equation (in dimensionless form)

$$m\frac{\partial^2\eta}{\partial t^2}+d\frac{\partial\eta}{\partial t}+D\frac{\partial^4\eta}{\partial x^4}-T\frac{\partial^2\eta}{\partial x^2}+K\eta=F \tag{1.6}$$

where m is the mass per unit area, d is the drag coefficient, D is the flexural rigidity, T is the longitudinal tension per unit width, K is the stiffness of the elastic base and F is the external force. In all the subsequent calculations, for simplicity, we shall neglect the longitudinal stress and stiffness of the elastic base that is, we shall put $T = K = 0$. The effect of these parameters will be indirectly estimated later.

In determining the force F acting on the plate, we shall assume that all the changes in the position of the plate occur solely due to the action of pressure and we shall neglect the action of shear stresses and the normal component of the viscous stresses. We must then put $F = -P(x, t)$.

System of equations (1.2) with condition (1.3) and boundary conditions (1.4) and (1.5) defines a non-linear boundary-value problem in a domain with moving walls for finding solutions that are periodic with respect to the x coordinate. The problem has the trivial solution $U = y$, $V = 0$, $P = 0$, which corresponds to the unperturbed boundary layer. We shall consider the perturbed solution of problem (1.2)–(1.5) by representing the solution in the form $U = y + u$, $V = \upsilon$, $P = p$ and assuming that the perturbations of u, υ and p are small. We shall study the stability of the boundary layer in the domain situated at a sufficient distance from the initial segment of the plate, where the perturbations can be treated as travelling waves.

It has been shown in [13] that the description of the stability of a boundary layer on a solid surface, based on three-deck theory, corresponds to the lower branch of the neutral stability curve at high Reynolds numbers. Our subsequent treatment will therefore correspond to the stability of a boundary layer with respect to long-wave perturbations.

2. THE LINEAR STABILITY PROBLEM

Searching for perturbations in the form of travelling waves, we represent the solution in the form

$$U = y - af'e^{ikx+\omega t},\quad v = ikafe^{ikx+\omega t},\quad p = ae^{ikx+\omega t},\quad \eta = \eta_0 e^{ikx+\omega t} \tag{2.1}$$

where a is the amplitude of the pressure, $f(y)$ is the amplitude of the stream function, k is the wave number, η_0 is the amplitude of the displacement of the surface over which the flow occurs, ω is the complex frequency of the oscillations, and a prime denotes differentiation with respect to y. The real part of the frequency $\omega = \omega_r + i\omega_i$ characterizes the increase ($\omega_r > 0$) or the attenuation ($\omega_r < 0$) of the perturbations with time.

We will also require that the arbitrary function $A(x, t)$ should satisfy the periodicity condition with respect to the x coordinate Condition (1.3) then leads to the equality

$$A(x,t) = \pm(a/k)e^{ikx+\omega t} \tag{2.2}$$

where the upper sign is taken when $k > 0$ and the lower sign when $k < 0$.

We now substitute expressions (2.1) into system of equations (1.2). Linearizing the first equation of (1.2) with respect to the amplitude a of the perturbations, we obtain an equation for the function $f(y)$

$$f''' - (\omega + iky)f' + ikf + ik = 0 \tag{2.3}$$

The second and third equations of system (1.2) are satisfied identically.

Linearizing boundary conditions (1.5) around the unperturbed surface we obtain the conditions for perturbations of the velocity [4]

$$u + \eta = 0,\ v = \partial\eta/\partial t \ \text{ when } \ y = 0$$

The boundary conditions on the wall for the stream function then take the form

$$f = \eta_0\omega/(ika),\ f' = \eta_0/a \ \text{ when } \ y = 0 \tag{2.4}$$

The limiting condition (1.4), where we have put $\lambda = 1$, gives, taking equality (2.2) into account

$$f' = \to \mp i/k \ \text{ when } \ y \to \infty \tag{2.5}$$

Boundary-value problem (2.3)–(2.5) differs from the problem formulated earlier in [11, 12] in the boundary conditions on the wall. When solving problem (2.3)–(2.5), we shall follow the known procedure [11]. We differentiate Eq. (2.3) and introduce the variable $z = \omega(ik)^{-2/3} + (ik)^{1/3}y$, where $|\arg z| < \pi/3$ when $y \to \infty$. As a result, we obtain the fourth-order equation

$$d^4f/dz^4 - zd^2f/dz^2 = 0 \tag{2.6}$$

The boundary conditions are

$$f = \eta_0\omega/(ika),\ df/dz = (\eta_0/a)(ik)^{-1/3},\ d^3f/dz^3 = -1 \ \text{ when } \ z = \omega(ik)^{-2/3} \tag{2.7}$$
$$df/dz \to \mp i(ik)^{-4/3} \ \text{ when } \ z \to \infty$$

We write the solution of Eq. (2.6) for the second derivative of the stream function f in the form

$$d^2f/dz^2 = c\mathrm{Ai}(z)$$

where Ai(z) is the Airy function, which decreases in the sector $|\arg z| < \pi/3$, and c is an arbitrary constant.

On satisfying the boundary conditions when $z = \omega(ik)^{-2/3}$, we obtain a solution in the form

$$f = -\left[\frac{d\,\mathrm{Ai}(\zeta)}{dz}\right]^{-1} \int_\zeta^z \int_\zeta^z \mathrm{Ai}(z_1)dz_1 dz + \frac{\eta_0}{a}(ik)^{-1/3}z, \quad \zeta = \omega(ik)^{-2/3}$$

The last of conditions (2.7) leads to the equation

$$-N(\zeta) + \frac{\eta_0}{a}k^{-1/3} = \mp k^{-4/3}, \quad N(\zeta) = e^{\pi i/6}\left[\frac{d\,\mathrm{Ai}(\zeta)}{dz}\right]^{-1} \int_\zeta^\infty \mathrm{Ai}(z)dz \tag{2.8}$$

In the limit when $\eta_0 \to 0$, Eq. (2.8) gives the dispersion equations obtained in [12] for the case of a solid wall. Note that, in the case of a solid wall, the quantity a is simply a multiplicative constant and does not appear in the dispersion equation.

We obtain an expression for the pressure from relation (2.8) in the form

$$a = \eta_0 k / [k^{4/3}N(\zeta) \mp 1] \tag{2.9}$$

We obtain the dispersion equation by equating the pressure in the fluid on the surface over which the flow occurs and the normal stresses which are developed in the flexible surface due to the action of the deformations $\eta = \eta_0 e^{ikx+\omega t}$. From Eq. (1.6), where we have put $T = K = 0$, we obtain

$$\eta_0(m\omega^2 + d\omega + Dk^4) = -a \tag{2.10}$$

The left-hand side of this equality is the sum of the inertial, frictional and elastic components.

Substituting expression (2.9) into Eq. (2.10) we obtain the dispersion equation

$$\frac{1}{N(\zeta)} = \pm k^{4/3} + \frac{k^{7/3}}{m\omega^2 + d\omega + Dk^4 \mp k} \tag{2.11}$$

The signs are chosen in accordance with the rule described above (see formula (2.2)). Next, we will investigate waves which propagate in the positive direction of the x axis and we shall therefore assume that $k > 0$ everywhere and, correspondingly, take the upper sign on the right-hand side of Eq. (2.11).

Equation (2.11) differs from the dispersion equation for the case of a solid wall [12] in that there is a second term on the right-hand side: It can be shown that the limiting case of a solid wall ($\eta_0 \to 0$) is equivalent to the requirement that the quantities m, D and d should tend to infinity.

The construction of the eigenvalue spectrum is well known in the case of a solid wall. The spectrum consists of a denumerable set of complex numbers which are determined, generally speaking, by the zeros of the Airy function [16]. A graphical representation of the behaviour of the first three roots accompanying a change in the wave number has been given in [12, 17]. The solution, which loses stability, is determined by the first root of the dispersion equation and, here, the values $\omega = -2.298i$, $k = 1.0005$ correspond to a neutral state [12].

3. NUMERICAL INVESTIGATION OF THE ROOTS OF THE DISPERSION EQUATION

The characteristic frequencies $\omega(k)$ were calculated from Eq. (2.11) for a fixed value of the shear stiffness $D = 10$ and values of the mass of the plate $m = 10, 4, 3, 2, 1$ and 0.1. The cases of a non-dissipative plate ($d = 0$) and plates with internal drag ($d = 0.5$ and 0.1) were considered. The behaviour of the roots of the dispersion equation in the complex plane $\omega = \omega_r + i\omega_i$ as a function of a change in the wave number k, which is a parameter, is shown in Fig. 1. The wave number values indicated for them correspond to the open circles.

The case of a drag-free plate ($d = 0$) is represented by the solid curves in Fig. 1(a). The curve for $m = 10$ differs only slightly from the corresponding curve for the case of a rigid wall (which is not shown in Fig. 1a). However, instability sets in at somewhat higher values of the wave number (when $k = 1.02$, we have $\omega = -0.0003-2.3228i$) compared with the case of a rigid wall ($k = 1.0005$). This suggests some

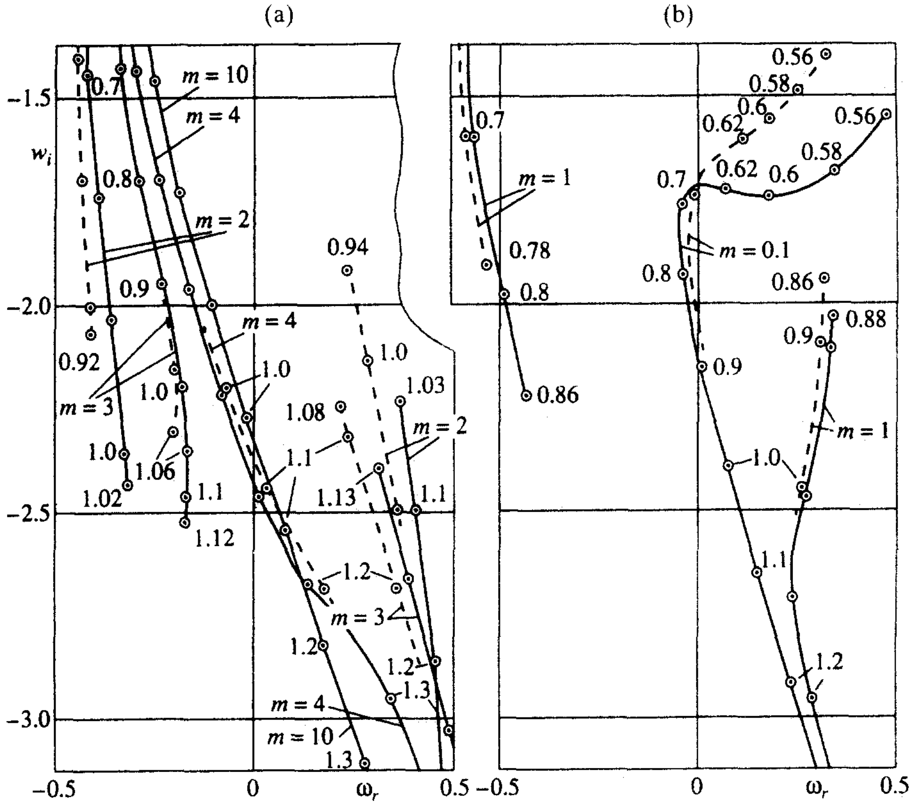

Fig. 1

stabilization of the flow when the wall is made compliant. When the mass of the plate is reduced ($m = 4$), the transition to an unstable state is even more delayed. For $m = 4$ when $k = 1.085$, we have $\omega = -0.00037-2.4204i$. However, in the unstable state, the rate of growth of the perturbations ω_r for values of the wave number greater than $k = 1.24$ is increased for $m = 4$ compared with $m = 10$.

Note that a reduction in the mass of the plate leads to an increase in the phase velocity of free flexural waves $c_p = (D/m)^{1/2}k$. It can be seen that an increase in c_p led to an increase in the wave number (by approximately 10%) corresponding to the neutral state. As was pointed out above, the three-deck boundary layer model which is used corresponds to the lower branch of the neutral curve in the case of high Reynolds number. However, this behaviour of the wave number, corresponding to the lower branch of the neutral curve, with respect to the velocity of free shear waves has been established in [7] in the case of not very high Reynolds numbers (of the order of $(2–4) \times 10^3$) close to the initial segment of the neutral curve. So, when the modulus of elasticity of the plate material was doubled, the wave number corresponding to the neutral state also increased by 10% ([7], Fig. 11).

The effect of the wall drag ($d = 0.5$) is shown by the dashed lines in Fig. 1(a). When $m = 10$, this effect is unimportant in practice but it is more noticeable when $m = 4$. It is seen that the frequency curve is displaced to the unstable state side. For example, a wave, which, when $k = 1.085$ and $d = 0$, was stable, loses stability when drag is introduced and the value of the frequency becomes equal to $\omega = 0.0132-2.4077i$. When $k > 1.3$, the wall drag practically ceases to affect the growth rates of the perturbations.

The introduction of wall drag therefore leads to destabilization of the flow. This behaviour of the perturbations corresponds to the well-known fact that internal wall drag destabilizes Tollmien–Schlichting waves which belong to waves of class A [3, 4, 7].

Calculations show that, in the stable state for $m = 3$ and 2, the attenuation rate of the perturbations is considerably greater than in the cases when $m = 10$ and $m = 4$. However, in the case when $k = 1.12$ ($\omega = -0.1737-2.5221i$), the curve for $m = 3$ terminates abruptly and continues when $k = 1.13$ from the point $\omega = 0.3109-2.3964i$ in the right-hand half-plane, and a significant growth rate of the perturbations is seen. The curve for $m = 2$ behaves in a similar manner but the discontinuity in the solution occurs earlier: when $k = 1.02$, we have $\omega = -0.3190-2.4280i$ and, when $k = 1.03$, we have $\omega = 0.3634-2.2346i$. Note that there is a reduction in the phase velocity of the wave $c = -\omega_i/k$ at the instant of time when the discontinuity occurs.

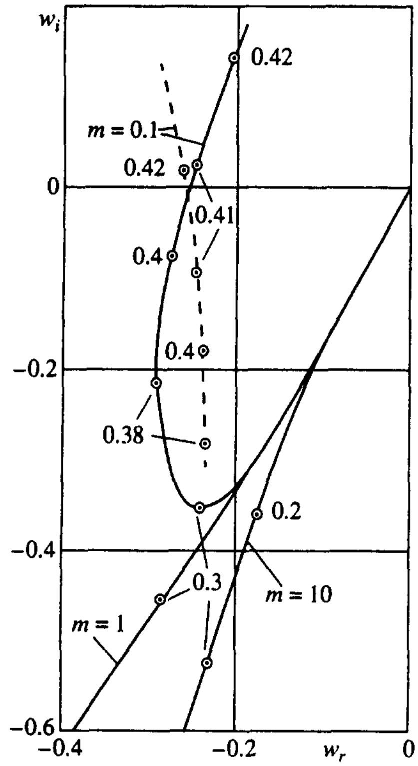

Fig. 2

It is seen from the graphs for $m = 3$ and $m = 2$ that, in the case of internal wall drag ($d = 0.5$), there are also discontinuities in the solutions but they occur at smaller values of the wave number, that is, instability sets in earlier. This involves a definite destabilization of the flow when wall drag is introduced, that is, the waves behave in the same way as class A waves. However, a positive effect is seen with respect to the rate of growth (attenuation) of the perturbations: wall drag stabilizes the flow in the stable and in the unstable state, but this property is characteristic of class B waves [7]. Note that the effect of wall drag when $m = 2$ and $m = 3$ is more pronounced than in the cases when $m = 4$ and $m = 10$.

The behaviour of the frequencies at mass values $m = 1$ and $m = 0.1$ is shown by the solid lines for $d = 0$ and the dashed lines for $d = 0.1$ in Fig. 1(b). When $m = 1$, the attenuation rate in the stable state ω_r is significantly greater than when $m = 2$ and $m = 3$. The discontinuity in the solution sets in when $k = 0.86$ ($\omega = -0.4337-2.2190i$). When $k = 0.88$, the curve, corresponding to $m = 1$, continues in the right-hand half-plane from the point $\omega = 0.3370-2.0283i$.

A characteristic feature was observed in the unstable state when $m = 1$: the frequency curve approaches the imaginary axis when $k = 1.1$. The discontinuity in the solution when $m = 1$ and $d = 0.1$ corresponds to the range of wave numbers $k = 0.78-0.86$.

When the mass is reduced to a value of $m = 0.1$, the frequency curve emerges, when $k = 0.56$, from the point $\omega = 0.4729-1.5474i$ and intersects the imaginary axis at $k \approx 0.65$ and $k \approx 0.89$, that is, a neutral state of perturbations appears. When $m = 0.1$ and $d = 0.1$, the frequency curve starts, when $k = 0.56$, from the point $\omega = 0.3201-1.3951i$. It is clear from a comparison with the curve for $m = 0.1$ and $d = 0$ that wall drag at wave numbers $k < 0.6$ decreases the growth rate of the perturbations and it increases them somewhat when $k > 0.6$.

The behaviour of the roots at small wave numbers is shown by the solid curves for $d = 0$ and the dashed line for $d = 0.1$ in Fig. 2. The frequency curves emerge from the point $\omega = 0$ and are located in the left half-plane as in the case of a rigid wall [12]. When $m = 0.1$ and $d = 0$, the frequency curve makes a loop and emerges into the upper half-plane where the solution abruptly terminates at approximately $k = 0.43$. Wall drag at low wave numbers has a weak effect on the behaviour of the frequencies for $m = 1-10$. When $m = 0.1$ and $d = 0.1$, it is seen that the wall drag, when $k < 0.41$, destabilizes the flow (a property of class A waves) and, when $k > 0.41$, it stabilizes the flow (class B waves).

So, when $m = 0.1$, conditions are possible when a wave travels with a low phase velocity or is completely arrested and changes into a standing wave. Such a state of motion is well known in aero-elasticity as "static divergence". It has been studied experimentally [18] and, moreover, slowly moving waves were only observed in a boundary layer in which turbulence had artificially been created.

Hence, depending on the ratio of the plate parameters, two different forms of loss of stability exist: when $m = 10$ and $m = 4$, the transition to unstable conditions is characterized by the onset of Tollmien–Schlichting waves and, when $m = 3, 2, 1$ or 0.1, there is no neutral oscillatory state and instability sets in abruptly.

Generally speaking, this behaviour is atypical of linear systems. Hence, further confirmation of the results which have been obtained is required. A superficially similar phenomenon was possibly observed in [7]. In that paper, it was noted that, for certain parameters of a flexible plate, a sharp change in the solutions was discovered for a very small change in the wave number or the Reynolds number and, also, that sometimes it was impossible to find a neutral oscillatory state [7, pp. 490 and 502]. It was suggested [7] that these effects are associated with the interaction of the modes, that is, similar eigenvalues are responsible for modes which correspond to waves of class A and class B. This question has not been studied in detail.

4. ANALYSIS OF THE FORMS OF LOSS OF STABILITY

It was established above that the discontinuous solutions have the properties both of class A and of class B waves. It is well known that the fluid viscosity is important in the generation and loss of stability of Tollmien–Schlichting waves (class A waves) while class B waves are mainly determined by the flexibility of the surface around which the flow occurs and can also exist in an inviscid flow [3, 4, 8]. It is also well known that instability of class B waves arises under the action, on the surface $\eta(x, t)$ over which the flow occurs, of the pressure component which is in phase with the wave inclination $\partial\eta/\partial x$ and is proportional to the second derivative of the velocity profile calculated in the critical layer, where the phase velocity of the wave is equal to the flow velocity [3, 8]. In the case being considered, that is, when the three-deck model is used, the velocity profile is linear and the generation of instability must occur by a different mechanism.

Moreover, in the boundary layer, class B waves propagate with a phase velocity $c = -\omega_i/k$ which is close to the velocity of free flexural waves $c_p = (D/m)^{1/2}k$. In the case being considered, for example when $m = 3$ and $k = 1$, the velocity of the wave $c = 2.20$ while $c_p = 1.82$, that is, there is a significant difference between the velocities. At the same time, the role of viscosity in the occurrence of discontinuous instability remains unexplained; hence we cannot attribute the discontinuous solutions to class B waves in the traditional sense. An energy treatment is necessary in order to gain a clearer idea of the character of the waves in question. We shall not discuss the solution of this problem but confine ourselves to examining the formal properties of the solutions which have been obtained.

4.1. *Neutral conditions.* We will now consider neutral conditions for a flow over a plate without internal drag. When $d = 0$, we represent dispersion equation (2.11) in the form (2.10)

$$m\omega^2 + Dk^4 = k/[1 - k^{4/3}N(\zeta)] \tag{4.1}$$

In the case of neutral conditions, the frequency is a pure imaginary quantity: $\omega = i\omega_*$, where $\omega_* > 0$, and this means that the left-hand side of Eq. (4.1) must be real. In order that Eq. (4.1) should be satisfied, the quantity $N(\zeta)$ must be real. We will find the values of $\zeta = \omega(ik)^{-2/3}$ for which this is possible. The corresponding values of ζ must lie on the half-line $\arg\zeta = -5\pi/6$. We put $\zeta = e^{-5\pi i/6}\zeta_0$, where $\zeta_0 > 0$. The results of a calculation of $N(\zeta) = N_r + iN_i$ as a function of the change in ζ_0 as well as the parameter when $\zeta_0 > 1.5$ are presented in Fig. 3. When $0 < \zeta_0 < 1.5$, the curve $N(\zeta)$ lies in the lower half-plane.

When $\zeta_0 > 6$, the curve $N(\zeta)$ practically merges with the real axis and tends asymptotically to zero when $\zeta_0 \to \infty$, remaining in the upper half-plane. This can be shown by considering the asymptotic expansion of the function $N(\zeta)$ for large values of ζ. In the sector $|\arg\zeta| < \pi$, we have the expansion

$$N(\zeta) \sim -\frac{e^{\pi i/6}}{\zeta}\left[1 - \frac{1}{\zeta^{3/2}} + \frac{9}{4}\frac{1}{\zeta^3} + O\left(\frac{1}{\zeta^{9/2}}\right)\right], \quad \zeta \to \infty \tag{4.2}$$

where the fractional powers of ζ take their principal values. Then, on the half-line $\arg\zeta = -5\pi/6$,

$$N(\zeta) \sim \zeta_0^{-1}[1 + e^{\pi i/4}\zeta_0^{-3/2} + O(\zeta_0^{-3})], \quad \zeta_0 \to \infty$$

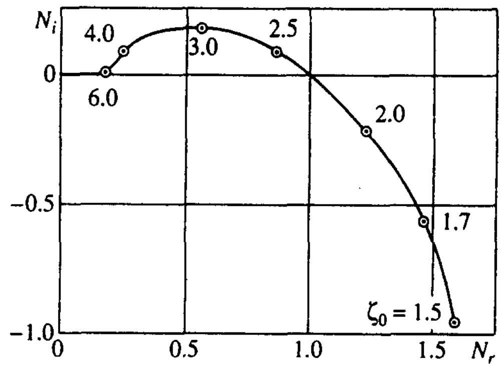

Fig. 3

Note that formula (4.2), retaining just the first two terms, gives an error of less than 1% when $\zeta_0 = 6$.

The calculations show that the curve $N(\zeta)$ intersects the real axis (that is, the function $N(\zeta)$ becomes real) at just a single point (when $\zeta \approx 2.2972e^{-5\pi i/6}$) and, at this point, $N(\zeta) \approx 0.99936-0.00001i \approx 1$.

It is easy to determine from this that, under neutral conditions, the frequency and wave number are connected by the relation $\omega = -2.2972k^{2/3}i$. This result only holds in the case of non-dissipative plates and, also, for the case of a rigid wall.

Note that the number $\alpha = 2.2972$, or a number which is close to it, is sometimes encountered in the analogous formulae in problems of hydrodynamic stability. This number is the argument for which the Tietjens function becomes real (see, for example, [13, 19]). It turned out that the function $N(\zeta)$ is closely associated with the above-mentioned Tietjens function.

4.2. *The non-existence of neutral conditions.* We will now find the conditions under which neutral conditions for the oscillations of a non-dissipative plate can exist. We will represent the frequency of the oscillations in the form $\omega = -\alpha k^{2/3}i$. With an accuracy which is sufficient for the purposes of this investigation, we put $N(\zeta) = 1$. Equation (4.1) then takes the form

$$-m\alpha^2 k^{4/3} + Dk^4 = k/(1-k^{4/3}) \tag{4.3}$$

that is, a relation is obtained which the wave number must satisfy under neutral conditions. This relation can be considered as an equation which determines, for a specified mass and stiffness of the plate, the wave number for which the effective stiffness of the plate (the left-hand side of the equation) compensates for the induced hydrodynamic pressure (the right-hand side).

Denoting the left-hand side of Eq. (4.3) by $Z_m(k)$ and the right-hand side by $F(k)$, we represent the graphical solution of Eq. (4.3) for $D = 10$ in Fig. 4. Neutral perturbations are determined by the points of intersection of the curves $Z_m(k)$ and $F(k)$. Having determined the wave number, it is then possible to find the frequency of the oscillations.

The curve $F(k)$ has a positive branch when $k < 1$ and a negative branch $F_-(k)$ when $k > 1$. When $m = 10$ and $m = 4$, there are two points of intersection of the curves $F_-(k)$ and $Z_m(k)$. When the mass of the plate is reduced, the intersection points merge and, subsequently, the intersection disappears ($m = 3, 1$). When the mass m is reduced further, an intersection of the curves again appears but now with the branch $F_+(k)$ when $k < 1$ ($m = 0.1$). We see that neutral oscillatory conditions do not exist for all ratios between the mass and stiffness of the plate. A reduction in the mass of the plate gives rise to a reduction in the absolute magnitude of the effective stiffness $Z_m(k)$. For a sufficiently small value of $Z_m(k)$, real values of the wave numbers, for which the effective stiffness can compensate for the induced pressure, do not exist and, consequently, neutral oscillatory conditions do not exist. The non-existence of neutral conditions indicates the possibility of the emergence of discontinuous solutions. However, it follows from the calculations presented in Section 3 that all the solutions when $m < 3$ are discontinuous and instability sets in abruptly.

It is clear that, taking account of the longitudinal stress T and the stiffness of the elastic base K in Eq. (4.1) would lead to a shift of the point where the curves $F_-(k)$ and $Z_m(k)$ touch in the direction of larger wave numbers. The appearance of discontinuous solutions would then be expected at larger wave numbers. However, the existence of a compressive stress ($T < 0$) would lead to the opposite effect.

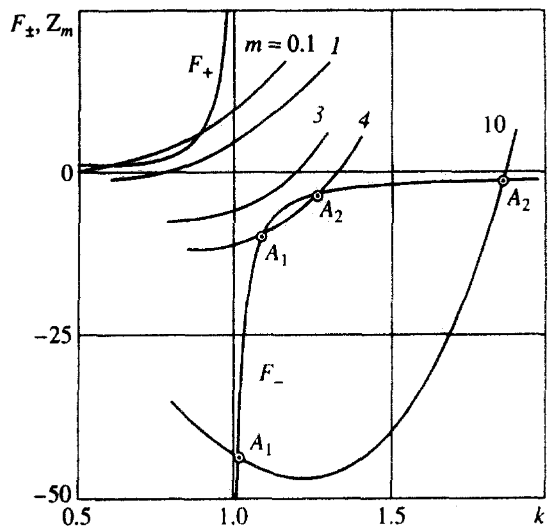

Fig. 4

The non-existence of neutral conditions for flow over a flexible surface at certain values of the parameters has been noted previously in [6].

4.3. *The uniqueness of the neutral solution.* The points of intersection of the curves $F_+(k)$ and $Z_m(k)$ when $m = 0.1$ correspond to the results presented in Fig. 1(b). When $m = 10$ and $m = 4$, the point of intersection A_1 of the curves $F_-(k)$ and $Z_m(k)$ in Fig. 4 corresponds to the neutral solution in Fig. 1(a). The existence of the point A_2 in Fig. 4 enables us to postulate the existence of another root of dispersion equation (4.1), that is, it can be postulated that there are two neutral waves which propagate with velocities $c_{1,2} = \alpha k_{1,2}^{-1/3}$. Then, when the mass of the plate is reduced, these two roots merge, that is, the waves overtake one another and interact, which leads to a situation similar to that which arises in the problem of the flutter of an infinite plate or in the case of Kelvin–Helmholtz instability [3, 4]. Effects associated with the non-existence of neutral perturbation conditions have been explained in [7] by a similar merging and interaction of modes.

In the case being considered here, the situation is different. One of the waves simply does not exist. We shall now show this to be so and assume that the points of intersection A_1 and A_2 of the curves $F_-(k)$ and $Z_m(k)$ correspond to two different roots of dispersion equation (4.1). Then, the point of contact of the curves must define a double root of Eq. (4.1). The frequency of the oscillations found here must satisfy Eq. (4.1), differentiated with respect to ω. We carry out this differentiation and represent the result in the form

$$e^{-\pi i/6}\,\mathrm{Ai}(\zeta)\left[\frac{d\,\mathrm{Ai}(\zeta)}{dz}\right]^{-1} = -\frac{2m\omega k^{1/3}}{(m\omega^2 + Dk^4)^2}\left[1 + \frac{\omega}{ik^2} - \frac{\omega}{ik(m\omega^2 + Dk^4)}\right]^{-1} \qquad (4.4)$$

It is seen that, under neutral conditions, when the frequency is a pure imaginary quantity, the right-hand side of Eq. (4.4) is pure imaginary. However, it has already been established (Section 4.1) that, under neutral conditions, the parameter ζ is must necessarily be equal to the quantity $\alpha e^{-5\pi i/6}$. The exact value of the left-hand side of Eq. (4.4) is at the same time equal to the complex quantity $0.40863-0.54438i$. It follows from this that Eq. (4.4), in general, does not have pure imaginary solutions, that is, the point of contact of the curves $F_-(k)$ and $Z_m(k)$ does not define a double root of the initial equation (4.1). Consequently, the point of intersection A_2 in Fig. 4 also does not define a root of Eq. (4.1) and corresponds to a spurious wave.

So, it has been established that there is no interaction between the waves in the case being considered and each wave develops individually.

4.4. *The asymptotic solution.* We will now consider the behaviour of the solutions of dispersion equation (4.1) in the case of large wave numbers. It follows from the calculations that the parameter $\zeta = \omega(ik)^{-2/3}$ increases as the wave number increases and $-\pi < \arg\zeta < -\pi/2$. We now make use of the asymptotic representation of the function $N(\zeta)$ for large ζ in the form of (4.2). Retaining the first two terms, we substitute expansion (4.2) into Eq. (4.1) and reduce it to the form

$$\omega = -ik^2\left(1 - \frac{1}{\zeta^{3/2}}\right) + \frac{k\omega}{m\omega^2 + Dk^4}$$

An approximate solution of this equation for large k and ζ is written in the form

$$\omega = -ik^2 + e^{\pi i/4} + O(k^{-1})$$

that is, the solution is located in the right-hand half-plane.

Hence, for large values of the wave number, the solutions of the dispersion equation (4.1) always define an unstable wave. However, it has been shown that, in the case of small wave numbers, the solutions of the dispersion equation lie in the left-hand half-plane (Fig. 2) and it has been shown in Section (4.2) that neutral perturbation conditions do not always exist. It follows from this that an abrupt loss of stability is possible for certain values of the plate parameters. The existence of discontinuous solutions can therefore be considered to be well founded.

5. CONCLUSION

We will now establish the physical reasons for the emergence of discontinuous solutions. We extrapolate the frequency values beyond the critical point $\omega(k_c)$ (that is, beyond the discontinuity point) and calculate the values of the amplitudes of the hydrodynamic pressure a and the stresses a_s which arise in the plate due to deformations

$$a = \frac{k}{k^{4/3}N(\zeta) - 1}, \quad a_s = m(\omega_i^2 - \omega_r^2) - Dk^4 - 2im\omega_r\omega_i$$

Specifying a wave number increment Δk, we carry out a frequency extrapolation using the formula

$$\omega(k) = \omega(k_c) + \frac{\partial\omega(k_c)}{\partial k}\Delta k, \quad k = k_c + \Delta k$$

Here k_c is the value of the wave number at the point where the solution terminates and $\partial\omega/\partial k = -G_k/G_\omega$, where G_k, G_ω are the derivatives with respect to k and ω of the dispersion relation $G(\omega, k) = 0$.

Calculations for $m = 2$ and $m = 3$ show that, when $k > k_c$, the real parts of the quantities a and a_s are different, $a_r > a_{sr} > 0$ and the imaginary parts are practically identical. Calculation of the phase shift φ, φ_s between the vertical velocity of the plate surface $v_s = \eta_0(\omega_r + i\omega_i)e^{ikx+\omega t}$ and the stresses $(a, a_s)e^{ikx+\omega t}$ shows that $\varphi_s < \varphi < \pi/2$ when $k > k_c$.

The following pattern of the development of instability is then possible. Under stable conditions, that is, when $\omega_r < 0$, the total energy of the perturbations of the fluid and the oscillations of the plate must tend to zero. If we have a non-dissipative plate, it must transfer its own energy to the fluid, where energy dissipation is possible either due to viscosity or due to the transfer of energy to the main flow by means of Reynolds stresses. For this to happen, the plate must perform work against the action of the pressure forces. The flow of energy, averaged over the wavelength, which is transferred by the plate to the fluid is given by the equation

$$\langle pv_s \rangle = -(1/2)\omega_r(m|\omega|^2 + Dk^4)$$

and, moreover, it must be positive. Its magnitude for a specified value of the frequency ω is regulated by the phase shift $\varphi < \pi/2$ and by the magnitude of the pressure components a_r, a_i. When $k < k_c$, such an interaction is possible and, at the point k_c, the plate can no longer provide the phase difference φ and the magnitude of the stresses a_s required for energy transfer. The elastic and inertial forces, which are developed by the plate, cannot compensate for the action of the pressure forces due to the fluid. In particular, when $m = 2$ or $m = 3$, the plate cannot counterbalance the pressure component a_r, which is in phase with the displacement of the surface, since the inertia of the plate is too small. As a result, when $k = k_c$, the plate "snaps", that is, the plate instantaneously changes its orientation and the flow becomes unstable.

Analogies of such behaviour in mechanical systems have been known for a long time. As an example, we can cite the problem of the snapping of weakly distorted elastic plates under the action of a distributed load, which was solved by Bubnov in 1902 [20].

Note that the abrupt instability which is being considered differs from the instability of class B waves that occurs under the action of a pressure component which is in phase with the slope of the wave.

REFERENCES

1. KRAMER, M.O., Boundary layer stabilization by distributed damping. *J. Am. Soc. Naval Engrs*, 1960, **72**, 25–33.
2. BENJAMIN, T. B., Effects of a flexible boundary on hydrodynamic stability. *J. Fluid Mech.*, 1960, **9**, Pt 4, 513–532.
3. BENJAMIN, T. B., The threefold classification of unstable disturbances in flexible surfaces bounding invisci flows. *J. Fluid Mech.*, 1963, **16**, Pt 3, 436–450.
4. LANDAHL, M. T., On the stability of a laminar incompressible boundary layer over a flexible surface. *J. Fluid Mech.*, 1962, **13**, Pt 4, 609–632.
5. KOROTKIN, A. I., Stability of a laminar boundary layer in an incompressible fluid on an elastic surface. *Izv. Akad. Nauk SSSR. MTT.*, 1966, 3, 39–44.
6. AMFILOKHIYEV, V. B., DROBLENKOV, V. V. and ZAVOROKHINA, A. S., Growth of small perturbations in a boundary layer on an elastic surface. *Zh. Prikl. Mekh. Tekh. Fiz.*, 1972, 2, 137–139.
7. CARPENTER, P. W. and GARRAD, A. D., The hydrodynamic stability of flow over Kramer-type compliant surfaces, Pt 1. Tollmien–Schlichting instabilities. *J. Fluid Mech.*, 1985, 155, 465–510.
8. CARPENTER, P. W. and GARRAD, A. D., The hydrodynamic stability of flow over Kramer-type compliant surfaces, Pt 2. Flow-induced surface instabilities. *J. Fluid Mech.*, 1986, **170**, 199–232.
9. RILEY, J.J., GAD-EL-HAK, M. and METCALFE, R.W., Compliant coatings. *Ann. Rev. Fluid Mech.*, 1988, 20, 393–420.
10. SAVENKOV, I. V., Suppression of the growth of non-linear wave packets by an elastic surface over which the flow occurs. *Zh. Vychisl. Mat. Mat. Fiz.*, 1995, **35**, 1, 95–103.
11. RYZHOV, O. S. and TERENT'YEV, Ye. D., An unsteady boundary layer with self-induced pressure. *Prikl. Mat. Mekh.*, 1977, **41**, 6, 1007–1023.
12. ZHUK, V. I. and RYZHOV, O. S., Free interaction and the stability of the boundary layer in an incompressible fluid. *Dokl. Akad. Nauk SSSR*, 1980, **253**, 6, 1326–1329.
13. SMITH, F. T., On the non-parallel flow stability of the Blasius boundary layer. *Proc. Roy. Soc. London. Ser. A.*, 1979, **366**, 1724. 91–109.
14. DUCK, P. W., The effect small surface perturbations on the pulsatile boundary layer on a semi-infinite flat plate. *J. Fluid Mech.*, 1988, **197**, 259–293.
15. SAVENKOV, I. V. and TERENT'YEV, Ye. D., Calculation of the longitudinal velocity in the linear problem of a vibrator in a subsonic boundary layer. *Izv. Akad. Nauk SSSR. MZhG*, 1987, 3, 61–67.
16. RYZHOV, O. S. and TERENT'YEV, Ye. D., Wave motions in a three-dimensional boundary layer. *Prikl. Mat. Mekh.*, 1991, **55**, 6, 912–927.
17. TERENT'YEV, Ye. D., The linear problem of a vibrator executing harmonic oscillations at subcritical frequencies in a subsonic boundary layer. *Prikl. Mat. Mekh.*, 1984, **48**, 2, 264–272.
18. GAD-EL-HAK, M., BLACKWELDER, R. F. and RILEY, J. J., On the interaction of compliant coatings with boundary-layer flows. *J. Fluid Mech.*, 1984, **140**, 257–280.
19. REID, W. H., The stability of parallel flows. *Basic Developments in Fluid Dynamics*, Edited by M. Holt, Vol. 1. Academic Press, New York, 1965, 249–307.
20. GRIGOLYUK, E. I. and LOPANITSYN, Ye. A., Finite deflection and snapping of thin elastic plate panels, *Prikl. Mat. Mekh.*, 1996, **60**, 5, 865–876.

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